SB-3578

M. A. / M. Sc. (Part-II) Examination

March / April – 2011

Mathematics : Paper - 5008

(Advanced Special Functions)

(Old Course)

Time : 3 Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ **M. A. / M. Sc. (Part-2)**

Name of the Subject :
☛ **Mathematics - 5008**

☛ Subject Code No. : **3 5 7 8** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) Attempt **all** questions.
(3) Figures to the right indicates marks.
(4) Notations and conventions are all standard.

1 (a) Show that the general solution of the hypergeomotriz differential equation $z(1-z)w''+\left[c-(a+b+1)z\right]w'-abw=0$ valid for $|z|<1$ is given by **7**

$$w=AF(a,\,b;\,c;\,z)+Bz^{1-c}F(a+1-c,b+1-c;\,2-c;\,z).$$

(b) If $|z|<1$ and $|z/(1-z)|<1$ show that **7**

$$F\begin{bmatrix} a,\,b; & \\ c; & z \end{bmatrix}=(1-z)^{-a}F\begin{bmatrix} a,\,c-b; & \dfrac{-z}{1-z} \\ b; & \end{bmatrix}$$

OR

1 (a) If $\left(a+b+\dfrac{1}{2}\right)$ is neither zero nor a negative integer and if $|x|<1$ and $|4x(1-x)|<1$ then prove that **7**

$$F\begin{bmatrix} a,\,b; & \\ a+b+\dfrac{1}{2}; & 4x(1-x) \end{bmatrix}=F\begin{bmatrix} 2a,\,2b; & \\ a+b+\dfrac{1}{2}; & x \end{bmatrix}$$

(b) Obtain the recurrene relation $\left[(1-x)\alpha_1+(A-B)x\right]F=$ **7**

$$(1-x)\alpha_1 F(\alpha_1+)-x\sum_{j=1}^{q}U_j F\left(\beta_j+\right),\ p=q+1.$$

2 (a) With usual notation prove that **8**

$${}_3F_2\left[\begin{matrix}a, & b, & c; \\ 1+a-b, & 1+a-c, & ;\end{matrix}\ 1\right]$$

$$=\frac{\left|\left(1+\frac{a}{2}\right)\right|(1+a-b)\left|(1+a-b)\right|(1+a-c)\left|\left(1+\frac{a}{2}-b-c\right)\right.}{\left|(1+a)\right|\left(1+\frac{a}{2}-b\right)\left|\left(1+\frac{a}{2}-c\right)\right|(1+a-b-c)}$$

(b) If $k(K)$ is the complete elliptic integral of the second kind, then prove that **6**

$$\int_0^t k\left(\sqrt{x(1-x)}\right)dx=\pi\sin^{-1}\left(\frac{t}{2}\right)$$

OR

2 (a) If n is a non negative integer and if b and c are independent of n, show **8**

that ${}_3F_2\left[\begin{matrix}-n, b, c; \\ 1-b-n, 1-c-n;\end{matrix}\ x\right]=(1-x)^n$

$${}_3F_2\left[\begin{matrix}\frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}, 1-b-c-n; \\ 1-b-n, 1-c-n;\end{matrix}\ \frac{-4x}{(1-x)^2}\right]$$

(b) Show that $\int_0^t x^{1/2}(t-x)^{-1/2}\left[1-x^2(t-x)^2\right]^{-1/2}dx$

$$=\frac{\pi}{2}t\,{}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{3}{4}; \\ 1;\end{matrix}\ \frac{t^4}{16}\right]$$

3 (a) Define orthogonality of a simple set of real polynomials $\phi_n(x)$. Show that the set $\phi_n(x)$ is orthogonal iff **6**

$$\int_a^b w(x)x^k\,\phi_n(x)dx=0,\quad k=0,1,\ldots,n-1;\ \text{for } a<x<b.$$

(b) Show that the function $w = {}_1F_1(a; bjz)$ is a solution of the differential equation $Zw'' + (b - z)w' - aw = 0$. **4**

(c) For the error function $erf(x)$. Show that $erf(x)$. **4**

$$erf(x) = \frac{2x}{\sqrt{\pi}}\ {}_1F_1\left(\frac{1}{3}; \frac{3}{2}; -x^2\right)$$

OR

3 (a) If $2a$ is not an odd integer. Prove that **6**

$${}_1F_1(a; 2a; 2z) = e^z\ {}_0F_1\left(-; a + \frac{1}{2}; \frac{z^2}{4}\right)$$

(b) State and prove the Kummer's first formula for the confluent hypergeometric function ${}_1F_1(a; b; z)$. **4**

(c) With usual notation prove that **4**

$$(b)_k \frac{d^k}{dz^k}\left[e^{-z}\ {}_1F_1(a; b; z)\right] = (-1)^k (b-a)_k\, e^{-z}\ {}_1F_1(a; b+k; z)$$

4 (a) Let the polynomials $f_n(x)$ are defined by **8**

$$(1-t)^{-c}\ \psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x)t^n \text{ where}$$

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n,\ \gamma_0 \neq 0 \text{ then show that}$$

$$f_n(x) = \frac{(c)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k (c+n)_k\ \gamma_k\ x^k}{\left(\frac{c}{2}\right)_k \left(\frac{c}{2} + \frac{1}{2}\right)_k}$$

(b) If $A(t)\ \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} y_n(x)t^n$ then prove that **6**

$$y_0^1(x) = 0,\ y_n^1(x) = y_{n-1}^1(x) - y_{n-1}(x) \text{ and}$$

$$y_n'(x) = -\sum_{k=0}^{n-1} y_k(x),\ n \geq 1$$

OR

4 (a) Define generating function. Let $\psi(u)=\sum_{n=0}^{\infty}\gamma_n u^n,\ \gamma_0\neq 0$ **8**

and the polynomials $f_n(x)$ defined by

$(1-t)^{-c}\,\psi\left(\frac{-4xt}{(1-t)^2}\right)=-\sum_{n=0}^{\infty}f_n(x),\ t^n$ then show that

$x f_n'(x)-n f_n(x)=-(c+n-1)\,f_{n-1}(x)-xf_{n-1}',\ n\geq 1$

(b) If $P_n(x)$ is defined by $A(t)\psi(xH(t))=\sum_{n=0}^{\infty}p_n(x)t^n$ **6**

with $\psi(t)=\sum_{n=0}^{\infty}\gamma_n t^n,\ A(t)=\sum_{n=0}^{\infty}a_n t^n$, $H(t)=\sum_{n=0}^{\infty}h_n t^{n+1}$

where $\gamma_0\neq 0,\ a_0\neq 0,\ h_0\neq 0$, then show that $P_n(x)$ is a polynomial in x of degree precisely n if $\gamma_n\neq 0$.

5 (a) Prove that **8**

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[p_n^{(\alpha,\beta)}(x)\right]^2=\frac{2^{\alpha+\beta+1}\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{n!\,(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)}$$

(b) Prove that **6**

$$\sum_{n=0}^{\infty}\frac{p_n^{(\alpha,\beta)}(x)t^n}{(1+\alpha)_n(1+\beta)_n}={}_0F_1\left[\begin{matrix}-; & \frac{t(x-1)}{2}\\ 1+\alpha; & \end{matrix}\right]{}_0F_1\left[\begin{matrix}-; & \frac{t(x+1)}{2}\\ 1+\beta; & \end{matrix}\right]$$

OR

5 (a) With usual notation prove that **8**

$$p_n^{(\alpha,\beta)}(x)=\frac{(-1)^n(1-x)^{-a}(1+x)^{-\beta}}{2^n\,n!}D^n\left[(1-x)^{n+\alpha}(1+x)^{n=\beta}\right]$$

(b) Prove that **6**

$$p_n^{(\alpha,\beta)}(x)=\frac{(-1)^n(1+\beta)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n\ ; & \frac{x+1}{2}\\ 1+\beta; & \end{matrix}\right]$$
