SB-3577/100

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M. A. / M. Sc. (Part-II) Examination

March / April – 2011

Mathematics : Paper - 5007

(Special Functions)

(Old Course)

Time : **3** Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fillup strictly the details of ☛ signs on your answer book.	Seat No. :
Name of the Examination : ☛ **M. A. / M. Sc. (Part-2)**	
Name of the Subject : ☛ **Mathematics - 5007 (Old)**	
☛ Subject Code No. : **3 5 7 7** ☛ Section No. (1, 2,.....) : **Nil**	Student's Signature

(2) Answer **all** questions.

(3) Figures to the right indicate marks of the question.

(4) Follow usual notations.

1 (a) When do you say that the infinite product converges to zero ? If no $a_n = -1$ then prove that $\prod_{n=1}^{\infty}(1+a_n)$ and $\sum_{n=1}^{\infty} Log(1+a_n)$ converge or diverge together. **5**

(b) Derive the integral representation for the function $_2F_1(a, b; c; z)$ **5**

(c) Show that $_2F_1\begin{bmatrix} -n, & b; & 1 \\ & c; & \end{bmatrix} = \dfrac{(-1)^n (1+a-c)n}{(c)n}$ **4**

OR

1 (a) State and prove the necessary condition for the convergence of the infinite product. Examine this condition for the product $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$. **5**

(b) Derive the $\theta-$form differential equation for the function ${}_2F_1(a, b;c;z)$ **5**

(c) Find the second order derivative of ${}_2F_1(a, b;c;x)$ with respect to x. **4**

2 (a) If $|z|<1$ and $|z/(1-z)|<1$ then show that **5**

$${}_2F_1\left[\begin{matrix}a, & b; & z\\ & c; & \end{matrix}\right]=(1-z)^{-a}\ {}_2F_1\left[\begin{matrix}a, & c-b; & -z/(1-z)\\ & c; & \end{matrix}\right]$$

Also sketch in z-plane, the region of validity of this transformation.

(b) Show that the Neumann polynomials $O_n(s)$, where $O_0(s)=s^{-1}$ are also given by **5**

$$O_n(s)=\frac{n}{4}\sum_{k=0}^{[n/2]}\frac{(n-k-1)!\left(2/s\right)^{n-2k+1}}{k!},\ n\geq 1$$

(c) Use Kummer's theorem for ${}_2F_1[-1]$ to evaluate : **4**

$${}_2F_1\left[\begin{matrix}1, & 1/2; & -1\\ & 3/2; & \end{matrix}\right].$$

OR

2 (a) If $2b$ is neither zero nor a negative integer and if $|y|<1/2$ and $|y/(1-y)|<1$ then prove that **5**

$$\left(1-y^{-a}\right)\ {}_2F_1\left[\begin{matrix}\frac{a}{2}, & \frac{a}{2}+\frac{1}{2}; & \frac{y^2}{(1-y)^2}\\ & b+\frac{1}{2}; & \end{matrix}\right]={}_2F_1\left[\begin{matrix}a, & b; & 2y\\ & 2b; & \end{matrix}\right]$$

(b) Show that (i) $J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin z$ and 5

(ii) $J_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cos z$

(c) Using Bessel's integral show that for real x and integral n, $|J_n(x)|\le 1$. 4

3 (a) Derive the Rodrigue's formula for the Legendre polynomial $P_n(x)$. 5

(b) Show that the Legendre polynomials are orthogonal with respect to the weight function unity over the interval $(-1, 1)$. 5

(c) Evaluate : $P_{2n}(0)$ and $P_{2n+1}(0)$. 4

OR

3 (a) Prove that the Legendre polynomial $P_n(x)$ is even function of x if n is an even positive integer and, is odd function of x if n is an odd positive integer. 5

(b) Derive Laplace's first integral formula for the Legendre polynomial. 5

(c) Determine the zeros of $P_3(x)$. 4

4 (a) Show that $\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)dx = 2^n n!\sqrt{\pi}$. 5

(b) Derive the differential equation for the Hermite polynomial. 5

(c) Obtain the series expansion $x^n=\sum_{k=0}^{[n/2]}\frac{n!\,H_{n-2k}(x)}{2^n\,k!\,(n-2k)!}$ 4

OR

4 (a) Show that $xH_n'(x)=nH_{n-1}'(x)+nH_n(x)$. 5

(b) Evaluate : $H_{2n+1}'(0)$ and $H_{2n}'(0)$. 5

(c) Obtain hypergeometric function form : **4**

$$H_n(x) = (2x)^n {}_2F_0\left(\frac{-n}{2}, \frac{-n}{2} + \frac{1}{2}; - - ; -\frac{1}{x^2}\right)$$

5 (a) Obtain the generating function relation : **5**

$$(1-t)^{-c} {}_1F_1\left[\begin{matrix} c; & -xt/(1-t) \\ 1+\alpha; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(c)_n}{(1+\alpha)_n} L_n^{(\alpha)}(x) t^n$$

(b) Show that $L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} D^n \left[e^{-x} x^{n+\alpha}\right]$ **5**

(c) Obtain the series for $L_3^{(\alpha)}(x)$. **4**

OR

5 (a) Derive the differential equation : **5**

$$xD^2 L_n^{(\alpha)}(x) + (1 - x + \alpha) DL_n^{(\alpha)}(x) + nL_n^{(\alpha)}(x) = 0.$$

(b) Show that the Leguerre polynomials are orthogonal with respect to weight function $w(r) = x^{\alpha} e^{-x}$ over the interval $(0, \infty)$ subject to the condition that $\text{Re}(\alpha) > -1$. **5**

(c) Show that $x^n = \sum_{k=0}^{n} \frac{(-1)^k \; n! \, (1+\alpha)_n}{(1+\alpha)_k \; (n-k)!} L_k^{(\alpha)}(x)$. **4**