SB-3536

M. Sc. (Part - II) Examination

March / April – 2011

Mathsmatics : Paper - 5021

(Computational Fluid Dynamics)

(New Course)

Time : Hours] [Total Marks :

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ **M. Sc. (Part - 2)**

Name of the Subject :
☛ **Mathsmatics : Paper - 5021**

☛ Subject Code No. : **3 5 3 6** ☛ Section No. (1, 2,.....) : **1&2**

Seat No. :

Student's Signature

(2) Attempt all questions.

(3) Figure to the right indicate marks.

(4) Follow usual notations and conventions.

1

(a) Derive the continuity equation (1.1) for the model of an infinitesimally small element fixed in space.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{\mathbf{V}}\right) = 0, \tag{1.1}$$

where, ρ = density of fluid element, and $\vec{\mathbf{V}}(u, v, w)$ = velocity of fluid element.

(b) Attempt any **2 out of 3**:

(i) Explain the term: Substantial Derivative.

Derive the equation for substantial derivative operator in vector notation as:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left(\nabla \cdot \vec{\mathbf{V}}\right),$$

where, $\vec{\mathbf{V}}(u, v, w)$ = velocity of fluid element.

(ii) Explain the terms:

1. Initial condition;
2. Boundary condition;

associated with partial differential equations (PDEs). Also explain the difference between Dirichlet and Neumann boundary conditions, by giving illustrations.

(iii) The governing equations for the irrotational, 2-dimensional, inviscid, steady flow of a compressible gas are given by

$$\begin{aligned} (1 - M_\infty^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} &= 0, \end{aligned} \tag{1.2}$$

where u and v are small perturbation velocities, measured relative to the free-stream velocity, and M_∞ is the freestream Mach number.

Use the eigenvalue method to classify the system of PDEs given by (1.2).

2

(a) Derive the energy equation (2.1) in conservation form for the model of an infinitesimally small fluid element moving with the flow.

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{V^2}{2}\right)\vec{\mathbf{V}}\right] \\ &= \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} \\ &+ \frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z} + \frac{\partial (v\tau_{xy})}{\partial x} + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z} \\ &+ \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y} + \frac{\partial (w\tau_{zz})}{\partial z} + \rho\vec{\mathbf{f}} \cdot \vec{\mathbf{V}}, \end{aligned} \tag{2.1}$$

where,

ρ = density of fluid element; τ = stress;

$\vec{\mathbf{V}}(u, v, w)$ = velocity of fluid element; e = internal energy;

∇ = Laplacian; T = temperature;

$\frac{V^2}{2}$ = kinetic energy; e = internal energy;

$\dot{q}$ = rate of volumetric heat addition per unit mass; k = constant;

$\vec{\mathbf{f}}$ = body force per unit mass acting on fluid element.

(b) Attempt any **2 out of 3**:

(i) Write points of differences between:

(I) Body forces and surface forces.

(II) Truncation error and round-off error (in a finite difference method).

(III) Quasilinear PDE and non-linear PDE.

(ii) Derive the second order central difference formula (2.2) for the mixed derivative.

$$\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{(i,j)} = \frac{u(i+1,j+1) - u(i+1,j-1) - u(i-1,j+1) + u(i-1,j-1)}{4\,\triangle x \triangle y} + O\left((\triangle x)^2, (\triangle y)^2\right) \quad (2.2)$$

(iii) Explain the various models for the continuum fluid used to derive the governing equations of fluid motion.

(a) The wave equation is a second-order hyperbolic PDE given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (3.1)$$

where c is a fixed constant equal to the propagation speed of the wave.

The *mid-point leapfrog* finite difference scheme to solve the PDE (3.1) is given by

$$u(i,j+1) = c^2\alpha^2\,[u(i+1,j) + u(i-1,j)] - u(i,j-1) + 2\,(1 - c^2\alpha^2)\,u(i,j) \quad (3.2)$$

Use the finite difference scheme (3.2) to solve the PDE (3.1) subject to initial and boundary conditions:

$$u(0,t) = u(1,t) = 0 \quad ; \quad t > 0$$

$$\frac{\partial u(x,0)}{\partial t} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad ; \quad 0 \le x \le 1$$

$$u(x,0) = \sin^3(\pi x) \quad ; \quad 0 \le x \le 1$$

Integrate up to 2 time levels, by taking $c^2 = 1$, $\alpha = \frac{k}{h}$, and step-lengths $h = 0.25$ and $k = 0.2$.

(b) Attempt any **2 out of 3**:

(i) The *relaxation method* used to solve the elliptic PDEs assumes that the function values on the boundary of a given shape are known. These boundary values are in turn used to compute the interior values.

Apply the *relaxation method* to solve the elliptic PDE

$$\nabla^2 u = 0,$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the *Laplace* operator and u is a scalar function,

subject to the square boundary defined by:

$$u(x,0) = \frac{1}{2}x^2; \qquad u(x,4) = x^2;$$

$$u(0,y) = 0; \qquad u(4,y) = 8 + 2y.$$

(ii) When is the finite difference scheme called stable?

Explain the stability of the *Crank-Nicholson* finite difference scheme, using matrix stability analysis.

(iii) Consider solving the one-dimensional parabolic heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.3}$$

subject to the initial and boundary conditions

$$u(x,0) = \begin{cases} 2x & ; \quad 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & ; \quad \frac{1}{2} \leq x \leq 1 \end{cases} \tag{3.4}$$

$$u(x+1,t) = u(x,t) \tag{3.5}$$

The *Laasonen* finite difference scheme to solve the parabolic PDE of the form (3.3) is given by

$$u(i,j) = -r\,u(i-1,j+1) + (1+2r)\,u(i,j+1) - r\,u(i+1,j+1), \tag{3.6}$$

where, $r = \dfrac{k}{h^2}$.

Apply the numerical scheme (3.6) to solve (3.3) subject to (3.4) and (3.5), by taking $c = 1$, step-lengths $h = 0.2$, $k = 0.2$, and hence find the value of $u(0.4, 0.2)$.

4

(a) Consider solving the two-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \tag{4.1}$$

The alternative direction explicit (ADE) finite difference scheme to solve the PDE of type (4.1) is given by the 5-point formula

$$\begin{aligned} u(i,j,k+1) &= (1-4\alpha)\,u(i,j,k) + \alpha\,[u(i+1,j,k) + u(i-1,j,k)] \\ &\quad + u(i,j+1,k) + u(i,j-1,k) \end{aligned} \tag{4.2}$$

Use (4.2) to solve the PDE (4.1) up to one time level, subject to the initial condition

$$u(x,y,0) = \sin \pi x \cdot \sin \pi y \quad ; \quad 0 \leq x, y \leq 1$$

and the condition

$$u(x,y,t) = 0 \quad ; \quad t > 0$$

on the boundaries. Take $h = \dfrac{1}{3}$ and $\alpha = \dfrac{1}{8}$.

(b) Attempt any **2 out of 3**:

(i) The Poisson's equation in two dimension is a PDE of elliptic type, given by

$$\nabla^2 u = f(x, y), \tag{4.3}$$

where ∇^2 is *Laplace* operator.

Form a mesh-grid with mesh-size $= 1$, and boundaries $x = y = 0$, $x = y = 3$, to solve the PDE (4.3), where $f(x, y) = -10(x^2 + y^2 + 10)$, subject to $u = 0$ on the boundary.

(ii) Explain the meaning of truncation error in finite difference methods.

Obtain the truncation error that is resulted when *Schmidt* method is applied to solve the one-dimensional heat equation (4.4)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{4.4}$$

(iii) Solve the following using *Galerkin*'s method.

$$\begin{aligned} \left[\frac{d^2u}{dx^2} - u\right] + x^2 &= 0 \quad ; \quad 0 < x < 1 \\ u(0) &= 1 \\ x \left.\frac{du}{dx}\right|_{x=1} &= 0. \end{aligned}$$

5

(a) The *Laplace*'s equation is a second order PDE of elliptic type, given by

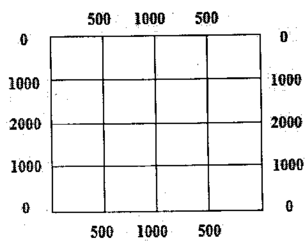
$$\nabla^2 u = 0, \tag{5.1}$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the *Laplace* operator and u is a scalar function.

The *five-point* finite difference scheme to solve the PDE (5.1) is given by

$$u(i, j) = \frac{1}{4}\left[u(i+1, j) + u(i-1, j) + u(i, j+1) + u(i, j-1)\right] \tag{5.2}$$

Use (5.2) to solve the PDE (5.1), with $h = k$ and the square-mesh domain defined with values as shown in the figure below.



(b) Attempt any **2 out of 3**:

(i) Derive the *Lax-Wendroff* formula (5.4) for solving the first order hyperbolic PDE (5.3).

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{5.3}$$

$$u(i, j+1) = u(i, j) - \frac{r}{2}\left[u(i+1, j) - u(i-1, j)\right] + \frac{r^2}{2}\left[u(i+1, j) - 2u(i, j) + u(i-1, j)\right], \tag{5.4}$$

where $r = \frac{k}{h}$ = step-ratio. Also give its graphical representation.

(ii) List out the drawbacks of finite difference methods (FDM) and state the procedure for solving field problems by FEM which overcomes the shortcomings of the FDM.

(iii) Explain by giving appropriate illustrations, the concepts of

(I) Connectivity matrix;

(II) Coordinate matrix;

(III) Boundary matrix,

in the study of finite element method (FEM).